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Comparison of Maximum Likelihood Estimation and Median Rank Regression Methods in Weibull Distribution Parameter Estimation

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Abstract

The Weibull distribution is widely used in reliability analysis and risk management due to its flexibility in modeling failure patterns. This study aims to compare two methods for estimating Weibull distribution parameters, namely Maximum Likelihood Estimation (MLE) from Median Rank Regression (MRR). The data used consists of simulation data with varying parameters and sample sizes, as well as case study data.shock absorber dataset from the library weibulltools containing failure time and censored data. Parameter estimation with MLE is performed using the Newton–Raphson algorithm, while MRR is performed through linear transformation and regression. Performance evaluation is performed using bias measures and Mean Squared Error (MSE) on simulated data, as well as Kolmogorov–Smirnov and Anderson–Darling tests on case study data.

Keywords: Weibull distribution, Maximum Likelihood Estimation, Median Rank Regression, Censored Data.

1. Introduction

Reliability analysis is one of the important fields in applied statistics, especially in studying the lifetime data of a component, system, or product. One of the most widely used probability distributions in reliability analysis is the Weibull distribution, due to its flexibility in representing various forms of failure data or failure times with relatively simple parameters. The Weibull distribution can describe increasing, decreasing, or constant failure rate patterns, making it relevant for many applications, ranging from manufacturing industries and engineering to the health sector.

To utilize the Weibull distribution optimally, accurate parameter estimation is required. The distribution's parameters will influence the shape of the probability density function, the cumulative distribution function, and other reliability measures. Therefore, the parameter estimation method becomes a crucial aspect in producing a distribution model that fits the empirical data.

There are various methods for estimating Weibull distribution parameters, including Maximum Likelihood Estimation (MLE) and Median Rank Regression (MRR). The MLE method is widely known for its consistent and efficient properties at large sample sizes; however, in practice, MLE can face computational difficulties or instability in results when sample sizes are limited. Meanwhile, the MRR method is often used as an alternative approach because it is relatively simple and provides reasonably good results in certain cases, especially in reliability data analysis.

Although both methods are widely used, there are differences in the characteristics, advantages, and limitations of each method. Therefore, a comparison between MLE and MRR in the context of Weibull parameter estimation becomes important to study, in order to provide a more comprehensive overview of the performance of each method.

Based on the description above, this research aims to compare the parameter estimation results of the Weibull distribution using the Maximum Likelihood Estimation and Median Rank Regression methods, so that the differences and advantages of each method in the application of reliability data analysis can be identified.

2. Literature Review

Distributions

A probability distribution is a function that describes the probability of a random value occurring within a specific range. In statistics, distributions are used to model random phenomena that occur in various fields, such as product reliability, economics, health, and the social sciences. Probability distributions are classified into two main types:

a) Discrete distributions, used when the random variable can only take specific, distinct values (e.g., Binomial, Poisson distributions).
b) Continuous distributions, used when the random variable can take any value within a continuous interval (e.g., Normal, Exponential, and Weibull distributions).

The selection of an appropriate distribution is crucial, as the form of the distribution will affect the analysis results, interpretation, and decision-making. Therefore, distribution parameter estimation becomes a fundamental step in applied statistics.

Parameter Estimation Method

Parameter estimation is the process of determining the value of population parameters based on sample data. Parameters in a probability distribution, such as the mean value, variance, or shape parameters, are usually not known with certainty and therefore need to be estimated.

In general, parameter estimation methods are divided into two major approaches:

a) Point Estimation Methods

Provides a single estimated value as a representation of the population parameter. Several widely used methods are:

[1] Method of Moments, introduced by Karl Pearson in the early 20th century, which bases estimation on equating population moments and sample moments.
[2] Maximum Likelihood Estimation (MLE), introduced by Ronald A. Fisher (1922), which is based on the principle of maximizing the likelihood function so that the chosen parameters are those most likely to produce the observed sample data. MLE is known to have consistent, efficient, and asymptotically normal properties.
[3] Rank Methods, which later developed into Median Rank Regression (MRR), are rooted in the concept of failure probability plotting in reliability analysis. This method uses a linear regression approach on transformed data based on the "median rank," which was introduced as one way to estimate the cumulative failure probability from sample data. This method is popular in engineering and industrial applications because it is simpler than MLE, especially for small sample sizes.

b) Interval Estimation Methods

Provides a range of values (confidence interval) that is expected to contain the true parameter value with a certain probability. In practice, each method has advantages and limitations. MLE is theoretically superior but computationally more complex, whereas rank-based methods like MRR are simpler and more practical, although their estimates are often less efficient for large samples.

Previous Research

Several previous studies have discussed the parameter estimation of the Weibull distribution using various methods. For example, Meeker & Escobar (1998) showed that the MLE method is superior for large sample sizes due to its efficiency properties. Meanwhile, Dodson (2006) emphasized that regression-based methods like MRR are more practical in industrial applications due to their ease of implementation.

These studies indicate that there are differences in the performance of the two methods under certain conditions. However, a more specific comparison between MLE and MRR for the Weibull distribution still needs to be conducted to gain a clearer understanding of the advantages of each method.

3. Materials and Methods

3.1. Materials

The data used in this study consists of two types: simulated data and case study data. The simulated data was generated from a Weibull distribution with various combinations of the shape parameter (β) and scale parameter (η), as well as small (n = 10 - 30), medium (n = 50 - 100), and large (n > 200) sample sizes. The use of simulated data aims to assess the performance of the estimation methods under controlled conditions, so that bias and Mean Squared Error (MSE) can be calculated directly by comparing the estimates to the true parameters.

Furthermore, this study also uses case study data in the form of the shock absorber dataset available in the weibulltools library in the R software. The dataset contains data on vehicle mileage until shock absorber failure, with some data being right-censored because the units were still functioning at the end of the observation period. The case study data was chosen to test the application of the estimation methods on real-world data frequently encountered in reliability analysis. The object of this research is the two-parameter Weibull distribution. This study focuses on estimating the shape (β) and scale (η) parameters using two approaches: Maximum Likelihood Estimation (MLE) and Median Rank Regression (MRR). The analysis was performed using R software (the weibulltools package) to process the case study data, and Python (the reliability package) to generate the simulated data. The MLE estimation process was carried out using the Newton–Raphson algorithm.

3.2. Methods

This research uses a comparative quantitative approach, namely a research approach that emphasizes numerical analysis with the main objective of comparing the performance of two Weibull distribution parameter estimation methods, namely Maximum Likelihood Estimation (MLE) from Median Rank Regression (MRR). The quantitative approach was chosen because parameter estimation and performance evaluation require objective numerical measures such as bias, Mean Squared Error (MSE), as well as the results of the distribution fit test. The comparative nature of this study is realized through testing the two estimation methods on different data conditions, both simulated data and case study data, as well as on variations in sample size and censoring levels.

Through this approach, the study not only assesses the accuracy of each method but also identifies conditions under which one method is superior to the other. Thus, the results are expected to provide a more comprehensive picture of the strengths and weaknesses of MLE and MRR, as well as their implications for selecting the appropriate estimation method for real-world cases in the fields of reliability and risk theory.

This method consists of several stages of analysis as follows:

3.2.1. Maximum Likelihood Estimation

Method Maximum Likelihood Estimation (MLE) is a parameter estimation approach introduced by Ronald A. Fisher in 1922. The basic principle is to choose distribution parameters that maximize the likelihood function, namely the probability of obtaining the observed sample data.

Suppose we have n events:

$$A_1, A_2, A_3, \dots, A_n$$

where$A_j$is the observation result for the j-th observation.

a) $A_j$can be a single data point or an interval
b) Intervals can appear if the data is grouped or there is data censorship.

Assume that the event $A_j$is the result of observing random variables X_jWhich can be seen in equation (3.1) below:

$$L(\theta) = \prod_{j=1}^{n} \Pr(X_j \in A_j | i) \tag{1}$$

and estimates maximum likelihood is the parameter vector that maximizes the likelihood function.

There is no guarantee that the likelihood function will have a maximum value for all possible parameter values. It is possible that if some parameters approach zero or infinity, the likelihood value will continue to increase. Therefore, care must be taken when maximizing this function, as local maximum may occur besides the global maximum. Often, it is not possible to maximize the likelihood function analytically (e.g., by reducing the partial derivatives to zero). Instead, a numerical approach is usually required.

Since the observations are assumed to be independent, the multiplication in the definition of likelihood represents the joint probability:

$Pr(X_1 \in A_1, \dots, X_n \in A_n | i)$,

This means that the likelihood function is the probability of obtaining a particular sample data, given specific assumptions. The estimate (MLE) is the parameter value that produces the model with the greatest probability of producing that data.

One of the main advantages of MLE is that this method is almost always applicable, because:

a) If you can write a probability equation based on the model, then you can calculate it with MLE.

b) On the other hand, if you cannot write and calculate probabilities based on a model, then there is no point in using that model, because it cannot be used to solve larger actuarial problems.

3.2.2. Median Rank Regression

Method Median Rank Regression (MRR) is a parameter estimation approach based on probability plotting which is widely used in reliability analysis. The concept of median rank was first introduced as a way to calculate the cumulative probability of failure in ranked data. The commonly used median rank formula is as follows:

$$F(t_i) = \frac{i - 0{,}3}{n + 0{,}4}, i = 1,2, \dots, n \tag{2}$$

with:

a) $F(t_i)$ is the cumulative failure probability on the 3rd data,

b) i is the ranking of data from the smallest,

c) n is the number of samples.

The transformed data is then plotted in Weibull coordinates so that the relationship between $ln[-ln\ (1 - F_t)\ )]$becomes linear. By performing a simple linear regression on these points, an estimate of the scale parameter is obtained α and β

The advantages of MRR are that the method is simple, intuitive, and relatively accurate in small sample sizes. However, its limitations include that the estimation results tend to be less efficient than MLE in large sample sizes and are dependent on the accuracy of data plotting.

3.2.3 Research Procedure

The research stages were carried out systematically as follows. First, simulation data was generated from the Weibull distribution with specific parameters and sample sizes, while case study data was processed by separating failure data from censored data. Second, parameter estimation was performed using the MLE method with the Newton–Raphson algorithm and the MRR method with linear regression. Third, the estimation results were compared by calculating the bias and MSE on the simulation data, as well as by testing the goodness-of-fit of the Kolmogorov–Smirnov and Anderson–Darling distributions on the case study data. The final stage was a comparative analysis of the results to determine the relative superiority of the two methods under different data conditions.

3.2.4 Evaluation and Output

Performance evaluation was conducted by calculating bias and MSE on simulated data, as well as by testing the distribution's suitability on case study data. The output of this study was the estimation of Weibull distribution parameters (β, η) using the MLE and MRR methods, the results of the performance comparison of the two methods, and recommendations for appropriate estimation methods for both complete and censored data.

4. Results and Discussion

4.1. Data Used

This study uses two types of data, namely simulation data and real case study data, to compare the performance of the methods. Maximum Likelihood Estimation (MLE) from Median Rank Regression (MRR) in the estimation of Weibull distribution parameters.

4.1.1. Simulation Data

The simulated data is generated from a Weibull distribution with the true parameters:

$$\beta = 2.0\ and\ \eta = 1000.$$

The resulting sample size consisted of 100 observations and was used to assess the performance of both methods by calculating bias and Mean Square Error (MSE). Examples of the first five data sets are shown in Table 4.1 below.

Table 4.1: Simulated Data from Weibull Distribution

No	Simulation Data
1	536.5
2	913.8
3	1548.2
4	867.9
5	1342.6

4.1.2. Real Case Study Data

Case study data is taken from package Weibulltools version 2.1.0 (Jockenhoevel et al., 2020), which contains reliability data for vehicle shock absorber components. This data has 38 observations with two variables:

- Distanceis the distance traveled (in kilometers) until the component fails or is sensored.
- Statusis the component condition (1 for failed, 0 for censored).

The entire data can be seen in Table 4.2 below.

Table 2: Component Failure Case Study DataShock Absorber

No	distance (km)	status
1	6700	1
2	6950	1
3	7820	1
4	9730	1
5	10200	1
6	11400	1
7	12600	1
8	12700	1
9	13600	1
10	13800	1
11	14400	1
12	15800	1
13	16000	1
14	17100	1
15	17500	1

16	18000	1
17	19000	1
18	19500	1
19	20000	1
20	21000	1
21	21500	1
22	22000	1
23	22500	1
24	23000	1
25	24000	1
26	24500	1
27	25000	1
28	25500	0
29	26000	0
30	26500	0
31	27000	0
32	27500	0
33	28000	0
34	28500	0
35	29000	0
36	29500	0
37	30000	0
38	30500	0

4.1.3 Case Study Data Description

Of the 38 observation data, 11 components failed (status = 1) and 27 components were censored (status = 0). Descriptive statistics for the datashock absorber is as follows:

Table 3: Descriptive Statistics of Component Failure Distance DataShock Absorber

Statistics	Value (km)
Minimum	6700
Q1	12675
Median	17520
Mean	17335
Q3	21800
Maximum	27490
Standard Deviation	6857

The visualization of the failure distance distribution is shown in Figure 4.1 below.

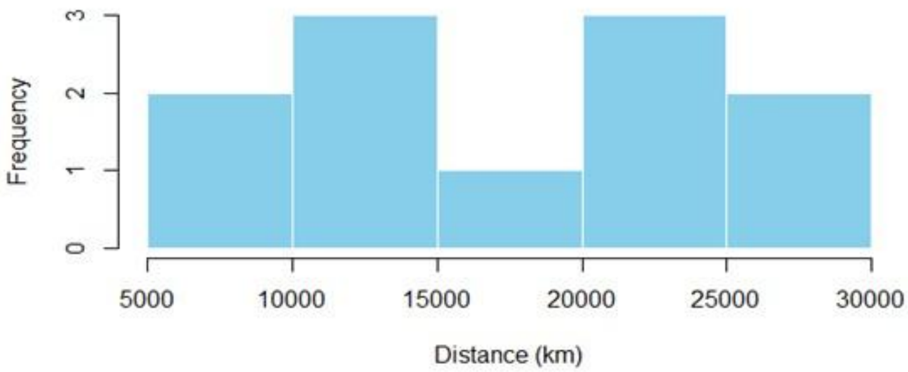

Figure 1: Failure Time Distribution

The graph shows that most failures occur in the 10,000 – 15,000 & 20,000 – 25,000 km range, with an increasing pattern at higher distances, which indicates a characteristic wear-out failure.

4.2. Weibull Distribution Parameter Estimation

The parameters of the Weibull distribution are estimated using two methods:

a) Maximum Likelihood Estimation (MLE) uses the function likelihood maximum.
b) Median Rank Regression (MRR) uses linear regression on median rank probability plot.

Estimations were performed for both types of data, and the results are summarized in Table 4.4 below.

Table 4: Weibull Distribution Parameter Estimation Results

Data Types	Method	b (Shape)	the (Scale)
Simulation	MLE	1927	948.57
Simulation	MRR	1919	944.89
Case study	MLE	1836	17321,24
Case study	MRR	1791	17004,56

4.3. Model Performance Evaluation

a. Simulation Data

For simulated data, model performance is evaluated by bias and Mean Square Error (MSE) from 100 repetitions.

Table 5: Weibull Distribution Parameter Estimation Results

Method	Bias(β)	Bias(n)	MSE(β)	MSE(n)
MLE	-0.073	-51.43	0.0125	2451.82
MRR	-0.081	-55.11	0.0141	2713.23

MLE has a bias value and MSE that is smaller than MRR, which means that the MLE estimation is more efficient and closer to the true parameters.

b. Case Study Data

For datashock absorber, a distribution suitability test was carried out using the Kolmogorov–Smirnov (K–S test).

```
        Exact one-sample Kolmogorov-Smirnov test

data:  failures
D = 0.12489, p-value = 0.987
alternative hypothesis: two-sided
```

The test results provide p-value > 0.05, indicating that the Weibull model fits the empirical data.

4.4. Comparative Analysis of Methods

Comparative analysis between methods Maximum Likelihood Estimation (MLE) from Median Rank Regression (MRR) is carried out to assess the differences in the results of the Weibull distribution parameter estimates, both in simulation data and case study data.

In general, the estimation results show that both methods produce relatively similar parameter values, with very small differences in the shape parameter (β) and scale parameter (η). For simulated data, the parameter values between the two methods differ only by $\Delta\beta \approx 0.008$ and $\Delta\eta \approx 3.7$. However, there are several important aspects that differentiate the performance of the two approaches.

The Maximum Likelihood Estimation (MLE) method shows superior performance compared to MRR in terms of efficiency and estimation accuracy. Theoretically, MLE has desirable statistical properties, such as:

a) Consistency, that is, the estimate will approach the true value as the sample size increases.
b) Efficiency, namely producing the minimum estimated variance among other unbiased estimators.
c) Asymptotically normal, thus allowing for more robust inferences through significance tests and confidence intervals.

Furthermore, the MLE method has the advantage of explicitly considering censored data in the estimation process, which is often found in reliability analysis or component lifespan data. This capability makes the MLE more robust and stable, especially for large sample sizes or data with a significant proportion of censored data (Genschel & Meeker, 2010).

Meanwhile, the Median Rank Regression (MRR) method has advantages in terms of implementation and visual interpretation. MRR uses a linear regression approach to median rank probability plot, making it easy to implement without requiring complex numerical optimization. MRR estimation results are often used in the exploratory phase because they provide a visual representation of the data's fit to the Weibull model. However, the main weakness of this method is its sensitivity to censored data and its inability to achieve maximum efficiency as in MLE (Christina Desriana et al., 2022). Thus, based on empirical results and theoretical considerations, the MLE method is recommended for reliability analysis involving censored data or large sample sizes, while the MRR can be used as an initial approach for rapid estimation and exploratory analysis.This conclusion is in line with the findings of Genschel and Meeker (2010), who stated that although the difference in estimates between MLE and MRR is generally small on complete data, the MLE method shows superior performance in terms of estimation stability and statistical efficiency across a range of data conditions.

4.5. Interpretation of Results

Based on the results of the Weibull distribution parameter estimation using the method*Maximum Likelihood Estimation*(MLE) from*Median Rank Regression*(MRR), it was found that both methods provided relatively consistent estimation results, both on simulation data and case study data.

In the simulation data, the difference in the parameter values β and η between the two methods is very small, which indicates that both are able to describe the data distribution pattern well. However, the bias value and *Mean Square Error (MSE)* shows that the MLE method produces estimates that are more efficient and closer to the true parameters than MRR. This indicates that MLE has a higher level of accuracy in estimating population parameters, especially at sufficiently large sample sizes.

Meanwhile, in the shock absorber case study data, the Weibull distribution was proven to be in accordance with the empirical data based on the results of the goodness-of-fit test (p-value > 0.05 in the Kolmogorov–Smirnov test). Parameter $\beta > 1$ show that the failure rate increases with increasing usage time, or in other words, the components experience wear-out failure. Scale parameter values η represents the characteristic time at which approximately 63.2% of the components are expected to have failed.

Overall, the results of this study strengthen previous findings by Genschel and Meeker (2010), that the MLE method provides more stable and efficient results than MRR. However, MRR remains relevant for use in initial analysis or when computational resources are limited, due to its simplicity and ability to provide reasonably good estimates under complete data conditions. From the results of this analysis, it can be concluded that the Weibull distribution is an appropriate model to describe shock absorber reliability data, and MLE is the recommended estimation method to obtain more accurate results in the context of censored data and large data.

5. Conclusion

Based on the results of the analysis and discussion that has been carried out, it can be concluded that the Weibull distribution is an appropriate model to describe mechanical component reliability data, especially in case study data.shock absorber. The results of the distribution suitability test using the method Kolmogorov–Smirnov shows that the p-value greater than 0.05, so there is insufficient evidence to reject the null hypothesis that the data are Weibull distributed. Thus, the Weibull distribution can be used appropriately to model the component failure pattern.

The results of parameter estimation using two methods, namely Maximum Likelihood Estimation (MLE) from Median Rank Regression (MRR), shows that both methods provide parameter estimates that are relatively close to each other. However, the MLE method produces more efficient and accurate estimates than MRR, both on simulated data and on case study data. The bias and Mean Square Error (MSE) the smaller MLE estimation results indicate that this method has higher accuracy relative to the true parameters. This finding supports the research of Genschel and Meeker (2010), which stated that MLE has better consistency and efficiency than MRR, especially on censored data.

However, the MRR method still has advantages in terms of ease of application and visual interpretation, because it uses a linear regression approach.Weibull probability plot This method is suitable for use in the exploratory phase to observe data distribution patterns. However, in the context of censored data, MRR is less than optimal because it doesn't directly account for sensor information in the estimation process.

In addition, the results of the shape parameter estimation (β) in the case study data shows a value greater than one ($\beta > 1$), which means that the failure rate increases with the time the component is used. This indicates that component failure is caused by the wear or aging process (wear-out failure). Meanwhile, the scale parameter (η) describes the characteristic time, namely the time when approximately 63.2% of the component units have failed. Overall, the MLE method is recommended for use in reliability analyses involving censored data and large sample sizes, while the MRR method can be utilized as an initial approach in exploratory analyses or when rapid estimation is required without complex numerical optimization. Both methods have their respective advantages and can complement each other in the context of reliability research and applications.

Based on the research results obtained, there are several suggestions that can be considered for further research development. First, future research is recommended to consider variations in sensor levels, such as: Type I, Type II, as well as random censoring, thus providing a deeper understanding of the robustness and stability of estimation results under varying data conditions. By incorporating sensor variations, the analysis can reflect more complex real-world conditions, as is often the case with industrial reliability data.

Further research could expand the approach by comparing other estimation methods, such as Bayesian estimation or L-moments, to obtain a more comprehensive picture of the performance of Weibull distribution parameter estimation methods. This approach also has the potential to provide more stable estimates on small samples or data with high censoring rates.

The application of this method can be extended to various types of empirical data from various industrial sectors, such as electronic systems, other automotive components, structural materials, or medical devices. This step is essential to test the model's generalizability and validity across different failure contexts.

It is recommended to develop software-based interactive visualizations, such as R Shiny or Python Dash, so that the results of reliability analysis can be interpreted and applied more easily by practitioners in the fields of engineering, manufacturing, and risk management. Thus, the results of this study are expected to contribute not only to the development of statistical methods in reliability analysis, but also to the practical application of data-driven decision-making in industrial environments.

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