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Solving The Fermat-Pell's Equation with Infinite Continuous Fractions

Mochamad Suyudi [1*]

[1] *Department of Mathematics, University of Padjadjaran. Jl. Ir. Soekarno KM 21 Jatinangor, Sumedang Indonesia*

**Corresponding author email: moch.suyudi@gmail.com*

Abstract

The special form $x^2 - dy^2 = \pm 1$ is called the Fermat-Pell's equation where d is a positive integer that is not a square. Let's say the x, y solution of this equation is a positive solution as long as x and y are both positive. Since solutions beyond $y = 0$ can be arranged in sets of four by sign combinations $\pm x, \pm y$, it is clear that all solutions will be known once all positive solutions are found. The result which gives us a starting point confirms that any pair of positive integers satisfying the Fermat-Pell's equation can be obtained from infinite continuous fraction denoting the irrational number $\sqrt{d}$.

Keywords: The Fermat-Pell's equation, infinite continuous fractions, irrational numbers

1. Introduction

Number theory is that branch of mathematics that is concerned with the properties of numbers. For this reason, number theory, which has a 4000 years of rich history, has traditionally been considered as pure mathematics. The theory of numbers has always occupied a unique position in the world of mathematics. This is due to unquestionable historical importance of the subject. It is one of the few disciplines having demonstrable results that predate the very idea of a university or an academy. The natural numbers have been known to us for so long that mathematician Leopold Kronecker once remarked, "God created the natural numbers, and all the rest is the work of man". Far from being a gift from Heaven, number theory has a long and sometimes painful evolution. (David M. Burton. 2011).

The theory of continued fractions begins with Rafael Bombelli, the last of great algebraists of Renaissance Italy. In his L'Algebra Opera (1572), Bombelli attempted to find square roots by using infinite continued fractions. One of the main uses of continued fraction is to find the approximate values of irrational numbers. (Gareth A. Jones and J. Mary Jones. 2007).

Srinivas Ramanujan has no rival in the history of mathematics. His contribution to number theory is quite significant. G.H.Hardy, commenting on Ramanujan's work, said "On this side (of Mathematics) most recently I have never met his equal, and I can only compare him with Euler or Jacobi".

Pell's equation $x^2 - dy^2 = 1$, was probably first studied in the case $x^2 - 2y^2 = 1$. Early mathematicians, upon discovering that $\sqrt{2}$ is irrational, realized that although one cannot solve the equation $x^2 - 2y^2 = 0$ in integers,one can at least solve the "next best things". The early investigators of Pell equation were the Indian mathematicians Brahmagupta and Bhaskara. In particulars Bhaskara studied Pell's equation for the values $d = 8,11,32,61,$ and 67 Bhaskara found the solution $x = 1776319049$, $y = 2261590$ for $d = 61$. (David M. Burton. 2011), (Neville Robbins. 2006).

Fermat was also interested in the Pell's equation and worked out some of the basic theories regarding Pell's equation. It was Lagrange who discovered the complete theory of the equation $x^2 - dy^2 = 1$. Euler mistakenly named the equation to John Pell. He did so apparently because Pell was instrumental in writing a book containing these equations. Brahmagupta has left us with this intriguing challenge: "A person who can, within a year, solve $x^2 - 92y^2 = 1$ is a mathematician." In general Pell's equation is a Diophantine equation of the form $x^2 - dy^2 = 1$, where d is a positive non square integer and has a long fascinating history and its applications are wide and Pell's equation always has the trivial solution $(x, y) = (1, 0)$, and has infinite solutions and many problems can be solved using Pell's equation. (David M. Burton. 2011), (Martin Erickson and Anthony Vazzana. 2010).

2. Methods

2.1 Linear Diophantine Equation

A linear equation which is to be solved for integers is called a Diophantine equation. The linear Diophantine equation of the form $ax + by = c$ has solution iff $(a, b)|c$.

Let $a, b, c \in$ Z. Consider the linear Diophantine equation $ax + by = c$.

a) If $(a, b) \nmid c$, there are no solutions.

b) If $(a, b)|c$, there are infinitely many solutions of the form

$$x = \frac{b}{d}k + x_0, y = -\frac{a}{d}k + y_0, \tag{1}$$

Where (x_0, y_0) is a particular solution and $k \in$ Z. (Neville Robbins. 2006).

Example: $6x + 9y = 21$

Since (6,9) = 3 and $3|21$, there are infinitely many solutions. By trial and error we find that, $x = -7, y = 7$ is a particular solution. Hence the general solution is given by

$$x = 3k - 7, y = -2k + 7$$

1.2 Infinite Continuous Fractions

If $[a_0;\ a_1, a_2, \ldots]$ is an infinite continuous fraction, The value of this infinite continuous fraction is $\lim_{k\to\infty} c_k$, where c_k is the k −th convergent value. To determine c_k, take $[a_0;\ a_1, a_2, \ldots]$ is an infinite continuous fraction. With

$$p_0 = a_0, q_0 = 1$$
$$p_1 = a_1 a_0 + 1, q_1 = a_1$$
$$p_k = a_k p_{k-1} + p_{k-2}, q_k = a_k q_{k-1} + q_{k-2}, k \geq 2,$$
$$c_k = \frac{p_k}{q_k}. \tag{2}$$

Let $[a_0;\ a_1, a_2, \ldots]$ is an infinite continuous fraction with $a_k > 0$ for $k \geq 1$. Then $[a_0;\ a_1, a_2, \ldots]$ is irrational. (David M. Burton. 2011).

Write $x = [a_0;\ a_1, a_2, \ldots]$. We will show that x is irrational. Suppose inversely that $x = \frac{p}{q}$, where p and q are integers. We will show that this leads to a contradiction.

Since the odd convergence is greater than x and the even convergence is less than x,

$$c_{2k+1} > x > c_{2k}.$$

Then

$$c_{2k+1} - c_{2k} > x - c_{2k} > 0,$$
$$\frac{(-1)^{2k}}{q_{2k}q_{2k+1}} > x - c_{2k} > 0,$$
$$\frac{1}{q_{2k}q_{2k+1}} > x - c_{2k} > 0,$$
$$\frac{1}{q_{2k}q_{2k+1}} > x - \frac{p_{2k}}{q_{2k}} > 0,$$
$$\frac{1}{q_{2k+1}} > xq_{2k} - p_{2k} > 0,$$
$$\frac{1}{q_{2k+1}} > xq_{2k} - p_{2k} > 0,$$
$$\frac{1}{q_{2k+1}} > \frac{pq_{2k}}{q} - p_{2k} > 0,$$
$$\frac{q}{q_{2k+1}} > pq_{2k} - p_{2k}q > 0.$$

Notice that this inequality is true for all k, and $pq_{2k} - p_{2k}q$ is an integer. But q is fixed(particular), and $q_{2k+1} \geq 2k + 1$, so if we make k big enough eventually q_{2k+1} will be bigger than q. Then $\frac{q}{q_{2k+1}}$ will be a fraction less than 1, and we have an integer $pq_{2k} - p_{2k}q$ is between 0 and a fraction less than 1. Since this is impossible, x cannot be rational.

We now know that every infinite continuous fraction made of positive integers represents an irrational number. The converse is also true, and gives an algorithm for calculating the expansion of a continuous fraction.

Let $x \in R$ be irrational. Suppose $x_0 = x$, and $a_k = [x_k]$, $x_{k+1} = \frac{1}{x_k - a_k}$ for $k \geq 0$.
Then for every $k \geq 0$,

$$x = [a_0; a_1, a_2, \ldots x_k] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots \cfrac{1}{a_{k-1} + \cfrac{1}{x_k}}}}$$

So, x_k represents the "infinite tail" of continuous fractions.
For $k = 0$, the claim is that $x = x_0$, which is true by definition.

Assume that the result holds for n. Then

$$x_{k+1} = \frac{1}{x_k - a_k}$$
$$x_k - a_k = \frac{1}{x_{k+1}}$$
$$x_k = a_k + \frac{1}{x_{k+1}}$$

Then

$$x = \left[a_0; a_1, a_2, \ldots, a_k + \frac{1}{x_{k+1}}\right] = [a_0; a_1, a_2, \ldots, a_k, x_{k+1}].$$

This the result for $k + 1$, so the result is true for $k \geq 0$ by induction.

Let $x \in R$ be irrational. Suppose $x_0 = x$, and $a_k = [x_k]$, $x_{k+1} = \frac{1}{x_k - a_k}$ for $k \geq 0$. Then for every $k \geq 0$, $x = [a_0;\ a_1, a_2, \ldots]$. (Rosen, Kenneth. H. 1984).

Step 1. x_k is irrational for $k \geq 0$.
Since x is irrational and $x_0 = x$, the result is true for $k = 0$.
Assume that $k > 0$ and that the result is true for $k - 1$. It will be shown that x_k is irrational.
Suppose on the contrary that $x_k = \frac{s}{t}$ where $s, t \in Z$. Then

$$\frac{s}{t} = \frac{1}{x_{k-1} - a_{k-1}} \text{ so } x_{k-1} = a_{k-1} + \frac{t}{s}.$$

Now all the $a_k's$ are clearly integers (since $a_k = [x_k]$ means they're outputs of the greatest integer function), so $a_{k-1} + \frac{t}{s}$ is the sum of an integer and a rational number. Therefore, it's rational, so x_{k-1} is rational, contrary to the induction hypothesis.
It follows that x_k is irrational. By induction, xk is irrational for all $k \geq 0$.
Step 2. The $a_k's$ are positive integers for $k \geq 1$.
Let $k \geq 0$. Since $a_k = [x_k]$, the definition of the greatest integer function gives

$$a_k \leq x_k \leq a_k + 1.$$

But x_k is irrational, so $a_k \neq x_k$. Hence,

$$a_k < x_k < a_k + 1,$$
$$0 < x_k - a_k < 1,$$
$$x_{k+1} = \frac{1}{x_k - a_k} > 1,$$
$$a_{k+1} = [x_{k+1}] \geq 1.$$

Since $k \geq 0$, this proves that the $a_k's$ are positive integers for $k \geq 1$.

Step 3.

$$\lim_{k\to\infty} c_k = \lim_{k\to\infty} [a_0;\ a_1, a_2, \ldots] = x.$$

First, We'll get a formula for x in terms of the $p's, q's$, and $a's$.
Then We'll find $\left|x - \frac{p_k}{q_k}\right|$ and show that it's less than something which goes to 0

Recall the recursion formulas for convergent:

$$p_k = a_k p_{k-1} + p_{k-2} \text{ dan } q_k = a_k q_{k-1} + q_{k-2}.$$

The right sides only involve terms up to a_k (and p's and q's of smaller indices). Therefore, the following fractions have the same p's and q's through index k:

$$[a_0;\ a_1, a_2, \dots, a_k, x_{k+1}] \text{ and } [a_0;\ a_1, a_2, \dots, a_k, x_{k+1}, \dots].$$

Using the preceding proposition and the recursion formula for convergent, We get

$$x = x_0 = [a_0;\ a_1;\ a_2; \dots; a_k; x_{k+1}] = \frac{x_{k+1}p_k + p_{k-1}}{x_{k+1}q_k + q_{k-1}}.$$

Therefore,

$$x - \frac{p_k}{q_k} = \frac{x_{k+1}p_k + p_{k-1}}{x_{k+1}q_k + q_{k-1}} - \frac{p_k}{q_k} = \frac{x_{k+1}p_k q_k + p_{k-1}q_k - x_{k+1}p_k q_k - p_k q_{k-1}}{(x_{k+1}q_k + q_{k-1})q_k}.$$

$$\frac{p_{k-1}q_k - p_k q_{k-1}}{(x_{k+1}q_k + q_{k-1})q_k} = \frac{(-1)^k}{(x_{k+1}q_k + q_{k-1})q_k}.$$

Take absolute values:

$$\left|x - \frac{p_k}{q_k}\right| = \frac{1}{(x_{k+1}q_k + q_{k-1})q_k}.$$

Now

$$x_{k+1} > [x_{k+1}] = a_{k+1}, \text{ so } x_{k+1}q_k + q_{k-1} > a_{k+1}q_k + q_{k-1} = q_{k+1}.$$

Therefore,

$$\frac{1}{x_{k+1}q_k + q_{k-1}} < \frac{1}{q_{k+1}},$$

$$\frac{1}{(x_{k+1}q_k + q_{k-1})q_k} < \frac{1}{q_{k+1}q_k},$$

$$\left|x - \frac{p_k}{q_k}\right| < \frac{1}{q_{k+1}q_k}$$

By an earlier lemma, $q_k \geq k$ and $q_{k+1} \geq k+1$, so

$$\left|x - \frac{p_k}{q_k}\right| < \frac{1}{q_{k+1}q_k} \leq \frac{1}{k(k+1)}.$$

Now $\lim_{k\to\infty} \frac{1}{k(k+1)} = 0$, so by the Squeezing Theorem

$$\lim_{k\to\infty} \left|x - \frac{p_k}{q_k}\right| = 0.$$

This implies that

$$\lim_{k\to\infty} \frac{p_k}{q_k} = x.$$

Example. Calculating continuous fraction expansion $\sqrt{23} \approx 4{,}8$. Consecutive irrational numbers x_k (and therefore integers $a_k = [x_k]$) can be calculated more easily, with the calculation shown below:

$$x_0 = \sqrt{23} = 4 + (\sqrt{23} - 4) \qquad a_0 = 4$$

$$x_1 = \frac{1}{x_0 - [x_0]} = \frac{1}{\sqrt{23} - 4} = \frac{\sqrt{23} + 4}{7} = 1 + \frac{\sqrt{23} - 3}{7} \qquad a_1 = 1$$

$$x_2 = \frac{1}{x_1 - [x_1]} = \frac{7}{\sqrt{23} - 3} = \frac{\sqrt{23} + 3}{2} = 3 + \frac{\sqrt{23} - 3}{2} \qquad a_2 = 3$$

$$x_3 = \frac{1}{x_2 - [x_2]} = \frac{2}{\sqrt{23} - 3} = \frac{\sqrt{23} + 3}{7} = 1 + \frac{\sqrt{23} - 4}{7} \qquad a_3 = 1$$

$$x_4 = \frac{1}{x_3 - [x_3]} = \frac{7}{\sqrt{23} - 4} = \sqrt{23} + 4 = 8 + (\sqrt{23} - 4) \qquad a_4 = 8$$

Since $x_5 = x_1$, also $x_6 = x_2$, $x_7 = x_3$, $x_8 = x_4$, then we get $x_9 = x_5 = x_1$, and so on, which means that blocks of integers 1, 3, 1, 8 repeat indefinitely. We find that the continuous fractional expansion of $\sqrt{23}$ is periodic with the form

$$\sqrt{23} = [4; 1,3,1,8,1,3,1,8, \dots]$$

$$= [4; \overline{1,3,1,8}]$$

Example. calculate the continuous fractional expansion and convergence of π.

$$x_0 = \pi, a_0 = [x_0] = [\pi] = 3$$

$$x_0 = \pi = 3 + (\pi - 3) \qquad a_0 = 3$$

$$x_1 = \frac{1}{x_0 - [x_0]} = \frac{1}{0.14159265\cdots} = 7.06251330\cdots \qquad a_1 = 7$$

$$x_2 = \frac{1}{x_1 - [x_1]} = \frac{1}{0.06251330\cdots} = 15.99659440\cdots \qquad a_2 = 15$$

$$x_3 = \frac{1}{x_2 - [x_2]} = \frac{1}{0.99659440\cdots} = 1.00341723\cdots \qquad a_3 = 1$$

$$x_4 = \frac{1}{x_3 - [x_3]} = \frac{1}{0.00341723\cdots} = 292.63467\cdots \qquad a_4 = 292$$

$$\vdots$$

Then calculating p_k, q_k, c_k, the results are as shown in the table below

Table 1: Expansion of the first five continuous fraction terms of π

x_k	a_k	p_k	q_k	c_k
π	3	3	1	$\frac{3}{1}$
7.06251…	7	22	7	$\frac{22}{7}$
15.99659…	15	333	106	$\frac{333}{106}$
1.00341…	1	355	113	$\frac{355}{113}$
292.63459…	292	103993	33102	$\frac{103993}{33102}$

So, the infinite continuous fraction for π is

$$\pi = [3; 7,15,1,292, \dots]$$

however, unlike the case of $\sqrt{23}$ where all the partial denominators of a_n are explicitly known, no pattern gives the complete ordering of a_n. The first five converge

$$\frac{3}{1}, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \frac{103993}{33102}$$

As an examination of Consequences and Theorems, note that we must have

$$\left|\pi - \frac{22}{7}\right| < \frac{1}{7^2}$$

now $\frac{314}{100} < \pi < \frac{22}{7}$, and therefore

$$\left|\pi - \frac{22}{7}\right| < \frac{22}{7} - \frac{314}{100} = \frac{1}{7\,.\,50} < \frac{1}{7^2}$$

as expected.

3. Results and Discussion

Consider the form of the Diophantine equation

$$x2 - d\,y2 = n. \tag{3}$$

If d is a perfect square, then the equation can be solved directly.

Example. Solve the Diophantine equation $x^2 - 9y^2 = 13$.

We can write the equation as $(x - 3y)(x + 3y) = 13$.

This is the equation in integers, and is the factorization of 13. There are only two ways to factor 13 in positive integers: 1·13 and 13·1. (we can check that negative factorization gives the same result.)

Let $x - 3y = 1$ and $x + 3y = 13$. It can be solved with

$$\begin{matrix} x & - & 3y & = & 1 \\ x & + & 3y & = & 13 \end{matrix} \text{ or } \begin{bmatrix} 1 & -3 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 13 \end{bmatrix}$$

Then

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 3 & 3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 13 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}.$$

$(x, y) = (7, 2)$ is an integer solution, so it qualifies as a solution to the initial equation. Since x and y appear as x^2 and y^2 in the original equation, $(-7, 2)$, $(7,-2)$, and $(-7,-2)$ are also solutions.

Similarly, $x - 3y = 13$ and $x + 3y = 1$ gives $(x, y) = (-7, 2)$ (which we already know).

So the solutions to the Diophantine equation $x2 - 9y2 = 13$ are (7, 2), (−7, 2), (7,−2), and (−7,−2).

Now suppose we change the problem to $x2 - 9y2 = 10$. Write it as $(x - 3y)(x + 3y) = 10$.

The possible factorizations of 10 are 1 · 10, 10 · 1, 2 · 5, and 5 · 2.

Try $x - 3y = 1, x + 3y = 10$.

Then

$$\begin{bmatrix} 1 & -3 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \end{bmatrix}, \quad so \quad \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 3 & 3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 10 \end{bmatrix} = \begin{bmatrix} \frac{11}{2} \\ \frac{3}{2} \end{bmatrix}$$

This is not an integer solution, so this factorization does not give an integer solution.

We can verify that the other factorizations do not give an integer solution. So, $x^2 - 9y^2 = 10$ has no integer solutions.

Now consider the case where d is not a perfect square. The following facts relate the solution to $x2 - d\,y2 = n$ with the continuous fraction expansion of $\sqrt{d}$.

Let $d > 0, d$ is not a perfect square, and $|k| < \sqrt{d}$. Every positive solution of $x^2 - d\,y^2 = k$ with $(x, y) = 1$ satisfies $x = p_n, y = q_n$ for some $n > 0$, where $\frac{p_n}{q_n}$ is the n-th convergence of the continuous fraction expansion of $\sqrt{d}$.

This statement does not say which convergence will give a solution. The special form $x^2 - d\,y^2 = \pm 1$ is called the Fermat-Pell equation. In this case, it is possible to say which convergence will solve the equation. Here will state the following facts, and give some examples.

First, recall from the theory of periodic continuous fractions that irrational squares in particular, a number of the form $\sqrt{d}$, where d is not squared have an extension of the periodic continuous fraction.

If $d > 0$ and d is not a perfect square, then the expansion of the continuous fraction of $\sqrt{d}$ is periodic, and has the form

$$[a_0; a_1, \ldots, a_{n-1}, 2a_0].$$

If $d > 0$, d is not a perfect square, and $|k| < \sqrt{d}$. Every positive solution of $x^2 - d\,y^2 = k$ with $(x, y) = 1$ satisfies $x = p_n, y = q_n$ for some $n > 0$, where $\frac{p_n}{q_n}$ is the n −th convergence of the continuous fraction expansion of $\sqrt{d}$.

Let t be the period of the expansion of $\sqrt{d}$.

(a) If t is even, then $x^2 - d\,y^2 = -1$ has no solutions. $x^2 - d\,y^2 = 1$ has solutions $x = p_{nt-1}, y = q_{nt-1}$ for $n \geq 1$.

(b) If t is odd, then $x^2 - d\,y^2 = -1$ has solutions $x = p_{nt-1}, y = q_{nt-1}$ for $n = 1, 3, 5, \ldots$, and $x^2 - d\,y^2 = 1$ has solutions $x = p_{nt-1}, y = q_{nt-1}$ for $n = 2, 4, 6, \ldots$.

Example. (a) Find the first 6 terms (a_0 through a_5) and the numerators and denominators of the first 6 convergents ((p_0, q_0) through (p_5, q_5)) of the continued fraction expansion of $\sqrt{14}$.
(b) Use the continued fraction for $\sqrt{14}$ to find solutions to the Fermat-Pell equations

$$x^2 - 14y^2 = -1 \text{ and } x^2 - 14y^2 = 1.$$

(a) After calculating $x_k, a_k, p_k,$ and q_k according to equation (2), the results are as shown in the table below

Table 2: Expansion of the first 6 terms continuous fraction of $\sqrt{14}$

x_k	a_k	p_k	q_k
3.74165…	3	3	1
1.34833…	1	4	1
2.87082…	1	11	3
1.14833…	1	15	4
6.74165…	6	101	27
1.34833…	1	116	31

So, the infinite continuous fraction for $\sqrt{14}$ is

$$\sqrt{14} = [3; 1,1,1,6,1, \ldots]$$
$$= [3; \overline{1,1,1,6}]$$

The first six converge is

$$\frac{3}{1}, \frac{4}{1}, \frac{11}{3}, \frac{15}{4}, \frac{101}{27}, \frac{116}{31}.$$

(b) The expansion has period 4, which is even. Hence, $x^2 - 14y^2 = -1$ has no solutions.
The first solution to $x^2 - 14y^2 = 1$ is (p_{4-1}, q_{4-1}) = (p_3, q_3) = (15, 4). We can check that

$$15^2 - 14 \cdot 4^2 = 1.$$

Example. (a) Find the first 6 terms (a_0 through a_5) and the numerators and denominators of the first 6 convergents ((p_0, q_0) through (p_5, q_5)) of the continued fraction expansion of $\sqrt{41}$.
(b) Use the continued fraction for $\sqrt{41}$ to find solutions to the Fermat-Pell equations

$$x^2 - 41y^2 = -1 \text{ and } x^2 - 41y^2 = 1.$$

(a) After calculating $x_k, a_k, p_k,$ and q_k according to equation (1), the results are as shown in the table below

Table 3: Expansion of the first 6 terms continuous fraction of $\sqrt{41}$

x_k	a_k	p_k	q_k
6.40312…	6	6	1
2.48062…	2	13	2
2.08062…	2	32	5
12.40312…	12	397	62
2.48062…	2	826	129
2.08062…	2	2049	320

So, the infinite continuous fraction for $\sqrt{41}$ is

$$\sqrt{41} = [6; 2,2,12,2,2, \ldots]$$
$$= [6; \overline{2,2,12}]$$

The first six converge is

$$\frac{6}{1}, \frac{13}{2}, \frac{32}{5}, \frac{397}{62}, \frac{826}{129}, \frac{2049}{320}.$$

(b) The period is 3, which is odd. The first solution to $x^2 - 41y^2 = -1$ is given by $(p_{3-1}, q_{3-1}) = (p_2, q_2)$ = (32, 5). We can check that

$$32^2 - 41 \cdot 5^2 = -1$$

For $x^2 - 41y^2 = 1$, We have 2 · 3 − 1 = 5, so the first solution is given by (p_5, q_5) = (2049, 320). We can check that

$$2049^2 - 41 \cdot 320^2 = 1.$$

In fact, we can generate the solution to the second equation using the solution to the first. Take (32, 5), and compute

$$(32 - 5\sqrt{41})^2 = 2049 + 320\sqrt{41}.$$

The coefficients (2049, 320) give the solution to the second equation.

4. Conclussion

Fermat-Pell equation where d is a positive integer that is not a square. The solution to this equation is a positive solution as long as x and y are both positive. it is clear that all solutions will be known once all positive solutions are found. The results confirm that any pair of positive integers that satisfy the Fermat-Pell equation can be obtained from a continuous fraction denoting the irrational number $\sqrt{d}$.

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